Announcements

1) Math Advising Session

   11:30 - 1:30 CB 2047
   (Math Library) - There
   will be pizza!

2) No online class

Back to Theorem:

Let $T \in \mathcal{T}^m(\mathbb{R}^n)$, $S \in \mathcal{T}^k(\mathbb{R}^n)$, $R \in \mathcal{T}^\ell(\mathbb{R}^n)$. Then

1) If $\mathrm{Alt}(S) = 0$, then

$$\mathrm{Alt}(T \otimes S) = 0.$$

2) $$\mathrm{Alt}(\mathrm{Alt}(T \otimes S) \otimes R)$$
$$= \mathrm{Alt}(T \otimes \mathrm{Alt}(S \otimes R))$$
$$= \mathrm{Alt}(T \otimes S \otimes R)$$ (associativity of Alt)

3) "$\wedge$" is associative, and

$$(T \wedge S) \wedge R$$
$$= T \wedge (S \wedge R)$$
$$= \frac{(k+m+l)!}{k!\,m!\,l!} \operatorname{Alt}(T \otimes S \otimes R)$$

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1) is on last class's notes - involves some group theory. Assuming this, let's prove 2) and 3).

2) Consider

$$T\otimes S - \mathrm{Alt}(T\otimes S) = Q$$

Then

$$\mathrm{Alt}(Q)$$

$$= \mathrm{Alt}(T\otimes S) - \mathrm{Alt}(\mathrm{Alt}(T\otimes S))$$

(Alt linear)

$$= \mathrm{Alt}(T\otimes S) - \mathrm{Alt}(T\otimes S)$$

$(\mathrm{Alt}(\mathrm{Alt}) = \mathrm{Alt})$

$$= 0.$$

So by 1),

$$0 = \mathrm{Alt}(Q \otimes R)$$

$$= \mathrm{Alt}\left(\left(T \otimes S - \mathrm{Alt}(T \otimes S)\right) \otimes R\right)$$

$$= \mathrm{Alt}\left(T \otimes S \otimes R - \mathrm{Alt}(T \otimes S) \otimes R\right)$$

$$= \mathrm{Alt}(T \otimes S \otimes R) - \mathrm{Alt}\left(\mathrm{Alt}(T \otimes S) \otimes R\right)$$

$$\Rightarrow \mathrm{Alt}(T \otimes S \otimes R)$$
$$= \mathrm{Alt}\left(\mathrm{Alt}(T \otimes S) \otimes R\right)$$

The equality

$$\mathrm{Alt}(T \otimes \mathrm{Alt}(S \otimes R)) = \mathrm{Alt}(T \otimes S \otimes R)$$

is proved similarly.

3) $(T \wedge S) \wedge R$

$$= \left(\frac{(m+k)!}{m!\,k!} \mathrm{Alt}(T \otimes S) \right) \wedge R$$

$$= \frac{(m+k)!}{m!\,k!} \left(\mathrm{Alt}(T \otimes S) \wedge R \right)$$

$$= \frac{\cancel{(m+k)!}}{m!\,k!} \frac{(m+k+\ell)!}{\cancel{(m+k)!}\,\ell!} \mathrm{Alt}\left(\mathrm{Alt}(T \otimes S) \otimes R \right)$$

$$= \frac{(m+k+\ell)!}{m!\,k!\,\ell!} \quad \mathrm{Alt}(T \otimes S \otimes R)$$

by 2)

The other equality

$$T \wedge (S \wedge R)$$

$$= \frac{(m+k+\ell)!}{m!\,k!\,\ell!} \operatorname{Alt}(T \otimes S \otimes R)$$

is proved similarly. $\square$

Proposition: (permutations)

Recall $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ is a 1-form,

$$\varphi_i\left(\sum_{j=1}^{n} a_j e_j\right) = a_i, \quad 1 \le i \le n.$$

1) $\varphi_i \wedge \varphi_i = 0$

$\forall \quad 1 \le i \le n$

2) If $i \neq j$,

$\varphi_i \wedge \varphi_j = -\varphi_j \wedge \varphi_i$

$1 \le i, j \le n$

Proof: Both at once!

$$\varphi_i \wedge \varphi_j = 2\,\mathrm{Alt}(\varphi_i \otimes \varphi_j).$$

$$(\varphi_i \wedge \varphi_j)(v_1, v_2)$$

$$= 2\,\mathrm{Alt}(\varphi_i \otimes \varphi_j)(v_1, v_2)$$

$$= \sum_{\sigma \in S_2} \mathrm{sign}(\sigma)\, \varphi_i(v_{\sigma(1)})\, \varphi_j(v_{\sigma(2)})$$

$= \varphi_i(v_1)\varphi_j(v_2)$ (identity)

$\quad - \varphi_i(v_2)\varphi_j(v_1)$ (flip)

Observe if $i=j$, we get

$$\varphi_i(v_1)\varphi_i(v_2)-\varphi_i(v_2)\varphi_i(v_1)$$

$$=0.$$

If $i\neq j$,

$$(\varphi_j\wedge\varphi_i)(v_1,v_2)$$

$$=\varphi_j(v_1)\varphi_i(v_2)-\varphi_j(v_2)\varphi_i(v_1)$$

$$=-(\varphi_i\wedge\varphi_j)(v_1,v_2). \quad \square$$

Theorem: $\Lambda^k(\mathbb{R}^n)$ has dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

A basis is given by

$$\{ \varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k} \}$$

where $1 \leq i_j \leq n \ \forall j$ and

$$i_1 < i_2 < \cdots < i_k .$$

Proof: Let $\omega \in \Lambda^k(\mathbb{R}^n)$.

We know $\omega \in \mathcal{T}^k(\mathbb{R}^n)$.

Then we can write

$$\omega = \sum_{i_1, i_2, \dots, i_k} c_{i_1, i_2, \dots, i_k} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \dots \otimes \varphi_{i_k}$$

$\forall \ 1 \le i_j \le n$ since we know

$\{\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k}\}$ is a basis for $\mathcal{T}^k(\mathbb{R}^n)$

Since $\omega \in \Lambda^k(\mathbb{R}^n)$, $\mathrm{Alt}(\omega) = \omega$. Then by linearity of Alt,

$$\omega = \mathrm{Alt}(\omega)$$

$$= \sum_{i_1, i_2, \dots, i_k} c_{i_1, i_2, \dots, i_k} \mathrm{Alt}(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \dots \otimes \varphi_{i_k})$$

Fix $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n\}$.

Consider $\mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$.

Now by our theorem,

$$\mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$$

$$= \frac{1}{k!}(\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k})$$

(Theorem allows us to drop all parentheses in wedge product).

<u>Case 1:</u> $\exists\ i_s, i_t,\ i_s = i_t$ $(s \neq t)$.

Using the second property of the proposition, we can produce

$$\varphi_{i_s} \wedge \varphi_{i_t}$$ in our

wedge product, which is then zero by the first property of the proposition since $i_s = i_t$.

This tells us that if

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \neq 0,$$

we must have

$$i_s \neq i_t \quad \text{for any}$$

$$1 \leq s, t \leq k.$$

We have then reduced the dimension of $\Lambda^k(\mathbb{R}^n)$

to no more than

$$n \cdot (n-1) \cdots (n-k) = P(n,k)$$

Case 2: $i_s \neq i_t$ for any $1 \leq s, t \leq k$.

Then by using property 2) from the proposition repeatedly, we may rearrange

$$\alpha_{i_1} \wedge \alpha_{i_2} \wedge - \wedge \alpha_{i_k} \text{ as }$$

$$\alpha_{j_1} \wedge \alpha_{j_2} \wedge - \wedge \alpha_{j_k}$$

Where

$j_1 < j_2 < \cdots < j_k$

and we may have to change the sign of the wedge product.

This says

$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ is a scalar multiple of

$\varphi_{j_1} \wedge \cdots \wedge \varphi_{j_k}$

This implies that

$$\dim(\Lambda^k(\mathbb{R}^n)) \leq \binom{n}{k}.$$

To show linear independence of $\{\varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k}\}$ with $i_1 < i_2 < \cdots < i_k$, apply to basis vectors.

(exactly as in the proof for $\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}$)

Observe that

$$\{ a_{i_1} \wedge \cdots \wedge a_{i_k} \}$$

$$i_1 < i_2 < \cdots < i_k$$

has cardinality $\binom{n}{k}$. $\square$

Orientation: From the theorem just proved,

$$\dim(\Lambda^n(\mathbb{R}^n)) = \binom{n}{n} = 1.$$

Since $\det \in \Lambda^n(\mathbb{R}^n)$, any $\omega \in \Lambda^n(\mathbb{R}^n)$ is a scalar multiple of det!

We could have

defined det as

the unique alternating

n-tensor on $\mathbb{R}^n$ satisfying

$$\det(e_1, e_2, \ldots, e_n) = 1.$$

Any basis for $\mathbb{R}^n$ can be obtained by applying an invertible linear operator to $\{e_1, \dots, e_n\}$. So if $\{f_1, \dots, f_n\}$ is such a basis and S is the linear operator,

$$\det(f_1, \dots, f_n)$$

$$= \det(S)\det(e_1, \dots, e_n)$$

$$= \det(S).$$

So we break bases up into two categories:

$\det(f_1, \dots, f_n) > 0$ and

$\det(f_1, \dots, f_n) < 0$.

An orientation on $\mathbb{R}^n$ is a choice of either bases with positive determinant or base with negative determinant but not both!